

# Two examples about zero torsion linear maps on Lie algebras <sup>\*</sup>

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## Abstract

The question of whether or not any zero torsion linear map on a non abelian real Lie algebra  $\mathfrak{g}$  is necessarily an extension of some  $CR$ -structure is considered and answered in the negative. Two examples are provided, one in the negative and one in the positive. In both cases, the computation up to equivalence of all zero torsion linear maps on  $\mathfrak{g}$  is used for an explicit description of the equivalence classes of integrable complex structures on  $\mathfrak{g} \times \mathfrak{g}$ .

## 1 Introduction.

Given a real Lie algebra  $\mathfrak{g}$ , the determination up to equivalence of zero torsion linear maps from  $\mathfrak{g}$  to  $\mathfrak{g}$  plays an important role in the computation of complex structures on direct products involving  $\mathfrak{g}$  ([2]). In the present note, we consider the question of whether or not any such zero torsion linear map for non abelian  $\mathfrak{g}$  is necessarily an extension of some  $CR$ -structure. We answer the question in the negative by computing (up to equivalence) all zero torsion linear maps from the real 3-dimensional Heisenberg Lie algebra  $\mathfrak{n}$  into itself. The result is then used to exhibit a complete set of representatives of equivalence classes of complex structures on  $\mathfrak{n} \times \mathfrak{n}$ . We also compute all zero torsion linear maps

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on  $\mathfrak{sl}(2, \mathbb{R})$ . In that case they are extensions of  $CR$ -structures. We deduce a complete set of representatives of equivalence classes of complex structures on  $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ .

## 2 Preliminaries.

Let  $G_0$  be a connected finite dimensional real Lie group, with Lie algebra  $\mathfrak{g}$ . A linear map  $J : \mathfrak{g} \rightarrow \mathfrak{g}$  is said to have *zero torsion* if it satisfies the condition

$$[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0 \quad \forall X, Y \in \mathfrak{g}. \quad (1)$$

If  $J$  has zero torsion and satisfies in addition  $J^2 = -1$ ,  $J$  is an (integrable) complex structure on  $\mathfrak{g}$ . That means that  $G_0$  can be given the structure of a complex manifold with the same underlying real structure and such that the canonical complex structure on  $G_0$  is the left invariant almost complex structure  $\hat{J}$  associated to  $J$  (For more details, see [3]). To any (integrable) complex structure  $J$  is associated the complex subalgebra  $\mathfrak{m} = \left\{ \tilde{X} := X - iJX ; X \in \mathfrak{g} \right\}$  of the complexification  $\mathfrak{g}_{\mathbb{C}}$  of  $\mathfrak{g}$ . In that way, (integrable) complex structures can be identified with complex subalgebras  $\mathfrak{m}$  of  $\mathfrak{g}_{\mathbb{C}}$  such that  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{m} \oplus \bar{\mathfrak{m}}$ , bar denoting conjugation.  $J$  is said to be abelian if  $\mathfrak{m}$  is. When computing the matrices of the zero torsion maps in some fixed basis  $(x_j)_{1 \leq j \leq n}$  of  $\mathfrak{g}$ , we will denote by  $ij|k$  ( $1 \leq i, j, k \leq n$ ) the torsion equation obtained by projecting on  $x_k$  the equation (1) with  $X = x_i, Y = x_j$ . The automorphism group  $\text{Aut } \mathfrak{g}$  of  $\mathfrak{g}$  acts on the set of all zero torsion linear maps and on the set of all complex structures on  $\mathfrak{g}$  by  $J \mapsto \Phi \circ J \circ \Phi^{-1} \quad \forall \Phi \in \text{Aut } \mathfrak{g}$ . Two  $J, J'$  on  $\mathfrak{g}$  are said to be *equivalent* (notation:  $J \equiv J'$ ) if they are on the same  $\text{Aut } \mathfrak{g}$  orbit. For complex structures and simply connected  $G_0$ , this amounts to the existence of an  $f \in \text{Aut } G_0$  such that  $f : (G_0, J) \rightarrow (G_0, J')$  is biholomorphic.

## 3 Case of $\mathfrak{sl}(2, \mathbb{R})$ .

Let  $G = SL(2, \mathbb{R})$  denote the Lie group of real  $2 \times 2$  matrices with determinant 1

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad ad - bc = 1. \quad (2)$$

Its Lie algebra  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$  consists of the zero trace real  $2 \times 2$  matrices

$$X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = xH + yX_+ + zX_-$$

with basis  $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $X_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$  and commutation relations

$$[H, X_+] = 2X_+, [H, X_-] = -2X_-, [X_+, X_-] = H. \quad (3)$$

Beside the basis  $(H, X_+, X_-)$ , we shall also make use of the basis  $(Y_1, Y_2, Y_3)$  where  $Y_1 = \frac{1}{2}H$ ,  $Y_2 = \frac{1}{2}(X_+ - X_-)$ ,  $Y_3 = \frac{1}{2}(X_+ + X_-)$ , with commutation relations

$$[Y_1, Y_2] = Y_3, [Y_1, Y_3] = Y_2, [Y_2, Y_3] = Y_1. \quad (4)$$

The adjoint representation of  $G$  on  $\mathfrak{g}$  is given by  $\text{Ad}(\sigma)X = \sigma X \sigma^{-1}$ . The matrix  $\Phi$  of  $\text{Ad}(\sigma)$  ( $\sigma$  as in (2)) in the basis  $(H, X_+, X_-)$  is

$$\Phi = \begin{pmatrix} 1 + 2bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{pmatrix}. \quad (5)$$

The adjoint group  $\text{Ad}(G)$  is the identity component of  $\text{Aut } \mathfrak{g}$  and one has

$$\text{Aut } \mathfrak{g} = \text{Ad}(G) \cup \Psi_0 \text{Ad}(G) \quad , \quad \Psi_0 = \text{diag}(1, -1, -1). \quad (6)$$

The adjoint action of  $G$  on  $\mathfrak{g}$  preserves the form  $x^2 + yz$ . The orbits are :

- (i) the trivial orbit  $\{0\}$ ;
- (ii) the upper sheet  $z > 0$  of the cone  $x^2 + yz = 0$  (orbit of  $X_-$ );
- (iii) the lower sheet  $z < 0$  of the cone  $x^2 + yz = 0$  (orbit of  $-X_-$ );
- (iv) for all  $s > 0$  the one-sheet hyperboloid  $x^2 + yz = s^2$  (orbit of  $sH$ );
- (v) for all  $s > 0$  the upper sheet  $z > 0$  of the hyperboloid  $x^2 + yz = -s^2$  (orbit of  $s(-X_+ + X_-)$ );
- (vi) for all  $s > 0$  the lower sheet  $z < 0$  of the hyperboloid  $x^2 + yz = -s^2$  (orbit of  $s(X_+ - X_-)$ ).

The orbits of  $\mathfrak{g}$  under the whole  $\text{Aut } \mathfrak{g}$  are, beside  $\{0\}$ :

- (I) the cone  $x^2 + yz = 0$  (orbit of  $X_-$ );
- (II) the one-sheet hyperboloid  $x^2 + yz = s^2$  (orbit of  $sH$ ) ( $s > 0$ );
- (III) the two-sheet hyperboloid  $x^2 + yz = -s^2$  (orbit of  $s(X_+ - X_-)$ ) ( $s > 0$ ).

**Lemma 1.** *Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ , and  $J : \mathfrak{g} \rightarrow \mathfrak{g}$  any linear map.  $J$  has zero torsion if and only if it is equivalent to the endomorphism defined in the basis  $(Y_1, Y_2, Y_3)$  (resp.  $(H, X_+, X_-)$ ) by*

$$J_*(\lambda) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \lambda & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda \in \mathbb{R} \quad , \quad (7)$$

$J_*(\lambda) \not\equiv J_*(\mu)$  for  $\lambda \neq \mu$

(resp.

$$J(\alpha) = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & \alpha & -\alpha \\ 1 & -\alpha & \alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}, \quad (8)$$

$J(\alpha) \not\equiv J(\beta)$  for  $\alpha \neq \beta$ ).

*Proof.* Let  $J = (\xi_j^i)_{1 \leq i, j \leq 3}$  in the basis  $(H, X_+, X_-)$ . The 9 torsion equations are in the basis  $(H, X_+, X_-)$ :

$$\begin{aligned} 12|1 & \quad 2(\xi_2^2 + \xi_1^1)\xi_2^1 + (\xi_2^2 - \xi_1^1)\xi_1^3 - (\xi_1^2 + 2\xi_3^1)\xi_2^3 = 0, \\ 12|2 & \quad 2(\xi_1^2\xi_2^1 + 1 + (\xi_2^2)^2) - \xi_1^3\xi_1^2 - 2\xi_2^3\xi_3^2 = 0, \\ 12|3 & \quad (\xi_1^3 + 2\xi_2^1)\xi_1^3 - 2(\xi_2^2 + 2\xi_1^1)\xi_2^3 + 2\xi_3^3\xi_2^3 = 0, \\ 13|1 & \quad (\xi_1^2 - 2\xi_3^1)\xi_1^1 + 2\xi_3^2\xi_2^1 + \xi_1^3\xi_3^2 - (\xi_1^2 + 2\xi_3^1)\xi_3^3 = 0, \\ 13|2 & \quad 2(\xi_2^2 - 2\xi_1^1)\xi_3^2 + (\xi_1^2 + 2\xi_3^1)\xi_1^2 - 2\xi_3^3\xi_3^2 = 0, \\ 13|3 & \quad \xi_1^3\xi_1^2 - 2\xi_1^3\xi_3^1 - 2 + 2\xi_2^3\xi_3^2 - 2(\xi_3^3)^2 = 0, \\ 23|1 & \quad 4\xi_3^1\xi_2^1 - 1 - \xi_2^2\xi_1^1 - \xi_2^3\xi_3^2 + (\xi_2^2 - \xi_1^1)\xi_3^3 = 0, \\ 23|2 & \quad 4\xi_3^2\xi_2^1 - (\xi_2^2 + \xi_3^3)\xi_1^2 = 0, \\ 23|3 & \quad 4\xi_2^3\xi_3^1 - (\xi_2^2 + \xi_3^3)\xi_1^3 = 0. \end{aligned}$$

$J$  has at least one real eigenvalue  $\lambda$ . Let  $v \in \mathfrak{g}$ ,  $v \neq 0$ , an eigenvector associated to  $\lambda$ . From the classification of the  $\text{Aut } \mathfrak{g}$  orbits of  $\mathfrak{g}$ , we then get 3 cases according to whether  $v$  is on the orbit (I),(II),(III) (in the cases (II), (III) one may choose  $v$  so that  $s = 1$ ).

Case 1. There exists  $\varphi \in \text{Aut } \mathfrak{g}$  such that  $v = \varphi(X_-)$ . Then, replacing  $J$  by  $\varphi^{-1}J\varphi$ , we may suppose  $\xi_3^1 = \xi_3^2 = 0$ . That case is impossible from 13|2 and 13|3.

Case 2. There exists  $\varphi \in \text{Aut } \mathfrak{g}$  such that  $v = \varphi(H)$ . Then we may suppose  $\xi_1^2 = \xi_1^3 = 0$ . Then from 12|2,  $\xi_3^2\xi_2^3 \neq 0$ , and 23|2, 23|3 yield  $\xi_2^1 = \xi_3^1 = 0$ . Then 12|3 and 13|2 successively give  $\xi_3^3 = \xi_2^2 + 2\xi_1^1$  and  $\xi_1^1 = 0$ . Now 12|2 and 23|1 read resp.  $-\xi_3^2\xi_2^3 + (\xi_2^2)^2 + 1 = 0$ , and  $\xi_3^2\xi_2^3 - (\xi_2^2)^2 + 1 = 0$ . Hence that case is impossible.

Case 3. There exists  $\varphi \in \text{Aut } \mathfrak{g}$  such that  $v = \varphi(X_+ - X_-)$ . Then we may suppose that  $v = X_+ - X_-$ . Now instead of the basis  $(H, X_+, X_-)$ , we consider the basis  $(Y_1, Y_2, Y_3)$ . The matrix of  $J$  in the basis  $(Y_1, Y_2, Y_3)$  has the form

$$J_* = \begin{pmatrix} \eta_1^1 & 0 & \eta_3^1 \\ \eta_1^2 & \lambda & \eta_3^2 \\ \eta_1^3 & 0 & \eta_3^3 \end{pmatrix}.$$

Then the 9 torsion equations  $*ij|k$  (the star is to underline that the new basis is in use) for  $J$  in that basis are:

$$\begin{aligned}
*12|1 & (\eta_1^3 + \eta_3^1)\lambda - (\eta_1^3 - \eta_3^1)\eta_1^1 = 0, \\
*12|2 & (\eta_1^1 + \lambda)\eta_3^2 - \eta_1^2\eta_3^3 = 0, \\
*12|3 & \eta_1^1\lambda - 1 + (\eta_3^1)^2 - (\eta_1^1 + \lambda)\eta_3^3 = 0, \\
*13|1 & \eta_3^2\eta_3^1 + \eta_1^2\eta_1^1 + \eta_3^2\eta_1^3 - \eta_1^2\eta_3^3 = 0, \\
*13|2 & \eta_1^1\lambda + 1 + (\eta_1^2)^2 + (\eta_3^2)^2 + \eta_1^3\eta_3^1 - (\eta_1^1 - \lambda)\eta_3^3 = 0, \\
*13|3 & \eta_3^2\eta_1^1 - \eta_1^2(\eta_3^1 + \eta_3^3) - \eta_3^2\eta_3^3 = 0, \\
*23|1 & \eta_1^1\lambda + 1 - (\eta_3^1)^2 + (\eta_1^1 - \lambda)\eta_3^3 = 0, \\
*23|2 & \eta_3^2\eta_3^1 - (\eta_3^3 + \lambda)\eta_1^2 = 0, \\
*23|3 & (\eta_1^3 + \eta_3^1)\lambda + (\eta_1^3 - \eta_3^1)\eta_3^3 = 0.
\end{aligned}$$

From  $*12|1$  and  $*23|3$ ,

$$\eta_1^1(\eta_1^3 - \eta_3^1) = -\eta_3^3(\eta_1^3 - \eta_3^1). \quad (9)$$

1) Suppose first that  $\eta_1^3 = \eta_3^1$ . Then  $\lambda\eta_1^3 = 0$ . 1.1) Consider the subcase  $\eta_1^3 = 0$ .  $*13|1$  and  $*13|3$  read resp.  $(\eta_3^3 - \eta_1^1)\eta_1^2 = 0$ ,  $(\eta_3^3 - \eta_1^1)\eta_3^2 = 0$ . Suppose  $\eta_3^3 \neq \eta_1^1$ . Then  $\eta_1^2 = \eta_3^2 = 0$ , and  $*13|2$  gives  $\eta_1^1\lambda + 1 = (\eta_1^1 - \lambda)\eta_3^3$ , which implies  $\eta_3^3 = 0$  by  $*23|1$ . As  $*12|3$  then reads  $1 = 0$ , this case  $\eta_3^3 \neq \eta_1^1$  is not possible. Now, the case  $\eta_3^3 = \eta_1^1$  is not possible either since then  $*23|1$  would read  $(\eta_1^1)^2 + 1 = 0$ . We conclude that the subcase 1.1) is not possible. Hence we are in the subcase 1.2)  $\eta_1^3 \neq 0$ . Then  $\lambda = 0$ . From  $*13|2$ ,  $\eta_3^3\eta_1^1 \neq 0$ . Then  $*23|1$  yields  $\eta_3^3 = \frac{-1+(\eta_1^1)^2}{\eta_1^1}$  and  $*13|2$  reads  $(\eta_1^1)^2 + (\eta_3^3)^2 + 2 = 0$ . This subcase 1.2) is not possible either. Hence case 1) is not possible, and we are necessarily in the case 2)  $\eta_1^3 \neq \eta_3^1$ . From (9),  $\eta_3^3 = -\eta_1^1$ . Then  $*13|2$  reads  $(\eta_1^1)^2 + (\eta_1^2)^2 + (\eta_3^2)^2 + 1 + \eta_1^3\eta_3^1 = 0$  hence  $\eta_1^3 \neq 0$  and  $\eta_3^1 = -\frac{(\eta_1^1)^2 + (\eta_1^2)^2 + (\eta_3^2)^2 + 1}{\eta_1^3}$ . From  $*12|2$ ,  $\eta_1^2 = \frac{\eta_3^2(\eta_1^1 + \lambda)}{\eta_1^3}$ . Then  $*23|2$  reads  $\eta_3^2((\eta_3^2)^2 + \lambda^2 + 1)(\eta_1^3)^2 + (\eta_1^1 + \lambda)^2(\eta_3^2)^2 = 0$ , i.e.  $\eta_3^2 = 0$ , which implies  $\eta_1^2 = 0$ . Now  $*12|1$  reads  $\lambda(1 + (\eta_1^1)^2 - (\eta_1^3)^2) = -\eta_1^1(1 + (\eta_1^1)^2 + (\eta_1^3)^2)$ . The subcase  $\eta_1^1 \neq 0$  is not possible since then  $*12|3$  would yield  $\lambda = -\frac{(\eta_1^1)^2 + (\eta_1^3)^2 - 1}{2\eta_1^1}$  and  $*12|1$  would read  $((\eta_1^1)^2 + (\eta_1^3 + 1)^2)((\eta_1^1)^2 + (\eta_1^3 - 1)^2) = 0$ . Hence  $\eta_1^1 = 0$ . Then  $*12|3$  reads  $(\xi_1^3)^2 = 1$ . The condition  $(\xi_1^3)^2 = 1$  now implies the vanishing of all the torsion equations. In that case

$$J_* = \begin{pmatrix} 0 & 0 & -\varepsilon \\ 0 & \lambda & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \quad \varepsilon = \pm 1.$$

Then in the basis  $(H, X_+, X_-)$

$$J = \begin{pmatrix} 0 & -\frac{\varepsilon}{2} & -\frac{\varepsilon}{2} \\ \varepsilon & \frac{\lambda}{2} & -\frac{\lambda}{2} \\ \varepsilon & -\frac{\lambda}{2} & \frac{\lambda}{2} \end{pmatrix}$$

The cases  $\varepsilon = \pm 1$  are equivalent under  $\Psi_0$ .  $\square$

**Remark 1.** Recall that a rank  $r$  ( $r \geq 1$ )  $CR$ -structure on a real Lie algebra  $\mathfrak{g}$  can be defined ([4]) as  $(\mathfrak{p}, J_{\mathfrak{p}})$  where  $\mathfrak{p}$  is some  $2r$ -dimensional vector subspace of  $\mathfrak{g}$  and  $J_{\mathfrak{p}} : \mathfrak{p} \rightarrow \mathfrak{p}$  is a linear map such that (a):  $J_{\mathfrak{p}}^2 = -1$ , (b):  $[X, Y] - [J_{\mathfrak{p}}X, J_{\mathfrak{p}}Y] \in \mathfrak{p} \quad \forall X, Y \in \mathfrak{p}$ , (c): (1) holds for  $J_{\mathfrak{p}}$  for all  $X, Y \in \mathfrak{p}$ . Then clearly  $J_*(\lambda)$  is an extension of a  $CR$ -structure.

## 4 Case of $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ .

We consider the basis  $(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}, Y_1^{(2)}, Y_2^{(2)}, Y_3^{(2)})$  of  $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ , with the upper index referring to the first or second factor. The automorphisms of  $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$  fall into 2 kinds: the first kind is comprised of the  $\text{diag}(\Phi_1, \Phi_2)$ ,  $\Phi_1, \Phi_2 \in \text{Aut } \mathfrak{sl}(2, \mathbb{R})$ , and the second kind is comprised of the  $\Gamma \circ \text{diag}(\Phi_1, \Phi_2)$ , with  $\Gamma$  the switch between the two factors of  $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ .

**Lemma 2.** Any integrable complex structure  $J$  on  $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$  is equivalent under some first kind automorphism to the endomorphism given in the basis  $(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}, Y_1^{(2)}, Y_2^{(2)}, Y_3^{(2)})$  by the matrix

$$\tilde{J}_*(\xi_2^2, \xi_5^2) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \xi_2^2 & 0 & 0 & \xi_5^2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -\frac{(\xi_2^2)^2+1}{\xi_5^2} & 0 & 0 & -\xi_2^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \xi_2^2, \xi_5^2 \in \mathbb{R}, \xi_5^2 \neq 0. \quad (10)$$

$\tilde{J}_*(\xi_2^2, \xi_5^2)$  is equivalent to  $\tilde{J}_*(\xi'^2_2, \xi'^2_5)$  under some first (resp. second) kind automorphism if and only if  $\xi'^2_2 = \xi_2^2$ ,  $\xi'^2_5 = \xi_5^2$  (resp.  $\xi'^2_2 = -\xi_2^2$ ,  $\xi'^2_5 = -\frac{(\xi_2^2)^2+1}{\xi_5^2}$ ).

*Proof.* Let  $J = (\xi_j^i)_{1 \leq i, j \leq 6} = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}$ ,  $(J_1, J_2, J_3, J_4 \text{ } 3 \times 3 \text{ blocks})$ , an integrable complex structure in the basis  $(Y_\ell^{(k)})$ . From lemma 1, with some first

kind automorphism, one may suppose  $J_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \xi_2^2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ ,  $J_4 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \xi_5^5 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ . As  $Tr(J) = 0$ ,  $\xi_5^5 = -\xi_2^2$ . Then one is led to (10) and the result follows (see [1], CSsl22.red and its output).  $\square$

**Remark 2.** The complex subalgebra  $\mathfrak{m}$  associated to  $\tilde{J}_*(\xi_2^2, \xi_5^2)$  has basis  $\tilde{Y}_1^{(1)} = Y_1^{(1)} - iY_3^{(1)}$ ,  $\tilde{Y}_1^{(2)} = Y_1^{(2)} - iY_3^{(2)}$ ,  $\tilde{Y}_2^{(2)} = -i\xi_5^2 Y_2^{(1)} + (1 + i\xi_2^2)Y_2^{(2)}$ . The complexification  $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$  of  $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$  has weight spaces decomposition with respect to the Cartan subalgebra  $\mathfrak{h} = \mathbb{C}Y_2^{(1)} \oplus \mathbb{C}Y_2^{(2)}$ :

$$\mathfrak{h} \oplus \mathbb{C}(Y_1^{(1)} + iY_3^{(1)}) \oplus \mathbb{C}(Y_1^{(2)} + iY_3^{(2)}) \oplus \mathbb{C}\tilde{Y}_1^{(1)} \oplus \mathbb{C}\tilde{Y}_1^{(2)}.$$

Then  $\mathfrak{m} = (\mathfrak{h} \cap \mathfrak{m}) \oplus \mathbb{C}\tilde{Y}_1^{(1)} \oplus \mathbb{C}\tilde{Y}_1^{(2)}$  with  $\mathfrak{h} \cap \mathfrak{m} = \mathbb{C}\tilde{Y}_2^{(2)}$ , which is a special case of the general fact proved in [5] that any complex (integrable) structure on a reductive Lie group of class I is regular.

## 5 Case of $\mathfrak{n}$ .

Let  $\mathfrak{n}$  the real 3-dimensional Heisenberg Lie algebra with basis  $(x_1, x_2, x_3)$  and commutation relations  $[x_1, x_2] = x_3$ .

**Lemma 3.** *Let  $J : \mathfrak{n} \rightarrow \mathfrak{n}$  any linear map.  $J$  has zero torsion if and only if it is equivalent to one of the endomorphisms defined in the basis  $(x_1, x_2, x_3)$  by:*

$$(i) \quad S(\xi_3^3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \xi_3^3 \end{pmatrix}, \quad \xi_3^3 \in \mathbb{R} \quad (11)$$

$$(ii) \quad D(\xi_1^1) = \begin{pmatrix} \xi_1^1 & 0 & 0 \\ 0 & \xi_1^1 & 0 \\ 0 & 0 & \frac{(\xi_1^1)^2 - 1}{2\xi_1^1} \end{pmatrix}, \quad \xi_1^1 \in \mathbb{R}, \xi_1^1 \neq 0 \quad (12)$$

$$(iii) \quad T(a, b) = \begin{pmatrix} 0 & -ab & 0 \\ 1 & b & 0 \\ 0 & 0 & \frac{ab-1}{b} \end{pmatrix}, \quad a, b \in \mathbb{R}, b \neq 0 \quad (13)$$

*Any two distinct endomorphisms in the preceding list are non equivalent.  $T(a, b)$  is equivalent to*

$$T'(a, b) = \begin{pmatrix} b & -b & 0 \\ a & 0 & 0 \\ 0 & 0 & \frac{ab-1}{b} \end{pmatrix} \quad (14)$$

*Proof.* Let  $J = (\xi_j^i)_{1 \leq i, j \leq 3}$  in the basis  $(x_1, x_2, x_3)$ . The 9 torsion equations are:

$$\begin{array}{ll}
12|1 & \xi_3^1(\xi_2^2 + \xi_1^1) = 0, \\
12|2 & \xi_3^2(\xi_2^2 + \xi_1^1) = 0, \\
12|3 & \xi_3^3(\xi_2^2 + \xi_1^1) - \xi_2^2\xi_1^1 + \xi_1^2\xi_2^1 + 1 = 0, \\
13|1 & \xi_3^2\xi_3^1 = 0, \\
13|2 & (\xi_3^2)^2 = 0, \\
13|3 & \xi_3^2(\xi_3^3 - \xi_1^1) + \xi_2^1\xi_3^1 = 0, \\
23|1 & (\xi_3^1)^2 = 0, \\
23|2 & \xi_3^2\xi_3^1 = 0, \\
23|3 & \xi_3^1(\xi_2^2 - \xi_3^3) - \xi_3^2\xi_2^1 = 0.
\end{array}$$

Hence  $\xi_3^1 = \xi_3^2 = 0$ , and we are left only with equation 12|3 which reads

$$\xi_3^3 \operatorname{Tr}(A) = \det(A) - 1 \quad (15)$$

where  $A = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix}$ . Suppose first  $\operatorname{Tr}(A) = 0$ . Then  $A^2 = -I$ , so that  $A$  is similar over  $\mathbb{C}$ , hence over  $\mathbb{R}$ , to  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Hence  $J \equiv \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ * & * & \xi_3^3 \end{pmatrix}$ . Now, since  $\xi_3^3$  does not belong to the spectrum of  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , taking the automorphism  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 1 \end{pmatrix}$  of  $\mathfrak{n}$  for suitable  $\alpha, \beta \in \mathbb{R}$ , one gets  $J \equiv S(\xi_3^3)$ . Suppose now  $\operatorname{Tr}(A) \neq 0$ . Then  $\xi_3^3 = \frac{\det(A)-1}{\operatorname{Tr}(A)}$ . If  $A$  is a scalar matrix, i.e.  $A = \xi_1^1 I$ , then  $J = \begin{pmatrix} \xi_1^1 & 0 & 0 \\ 0 & \xi_1^1 & 0 \\ * & * & \frac{(\xi_1^1)^2-1}{2\xi_1^1} \end{pmatrix} \equiv D(\xi_1^1)$ . If  $A$  is not a scalar matrix, then  $A$  is similar to  $\begin{pmatrix} 0 & -ab \\ 1 & b \end{pmatrix}$  for some  $a, b \in \mathbb{R}$ , and  $b \neq 0$  from the trace. Then  $J \equiv T(a, b)$ . Finally,  $T'(a, b) \equiv T(a, b)$  since the matrices  $\begin{pmatrix} 0 & -ab \\ 1 & b \end{pmatrix}$  and  $\begin{pmatrix} b & -b \\ a & 0 \end{pmatrix}$  are similar for they have the same spectrum and are no scalar matrices.  $\square$

**Remark 3.**  $S(\xi_3^3)$  is an extension of a rank 1  $CR$ -structure, however  $D(\xi_1^1), T(a, b)$  are not.

## 6 $CR$ -structures on $\mathfrak{n}$ .

**Lemma 4.** (i) Any linear map  $J : \mathfrak{n} \rightarrow \mathfrak{n}$  which has zero torsion and is an extension of a rank 1  $CR$ -structure on  $\mathfrak{n}$  such that  $\mathfrak{p}$  is nonabelian is equivalent to a unique

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \xi_3^3 \end{pmatrix}, \quad \xi_3^3 \in \mathbb{R}. \quad (16)$$

(ii) Any linear map  $J : \mathfrak{n} \rightarrow \mathfrak{n}$  which is an extension of a rank 1 CR-structure on  $\mathfrak{n}$  such that  $\mathfrak{p}$  is abelian is equivalent to a unique

$$\begin{pmatrix} \xi_1^1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \xi_1^1 \in \mathbb{R}. \quad (17)$$

$J$  has nonzero torsion.

*Proof.* For any nonzero  $X \in \mathfrak{p}$ ,  $(X, J_{\mathfrak{p}}X)$  is a basis of  $\mathfrak{p}$ . In case (i),  $[X, J_{\mathfrak{p}}X] \neq 0$ , since  $\mathfrak{p}$  is non abelian. Then  $[X, J_{\mathfrak{p}}X] = \mu x_3$ ,  $\mu \neq 0$ , and  $x_3 \notin \mathfrak{p}$  since otherwise  $\mathfrak{p}$  would be abelian. One may extend  $J_{\mathfrak{p}}$  to  $\mathfrak{n}$  in the basis  $(X, J_{\mathfrak{p}}X, \mu x_3)$  as

$$J = \begin{pmatrix} 0 & -1 & \xi_3^1 \\ 1 & 0 & \xi_3^2 \\ 0 & 0 & \xi_3^3 \end{pmatrix} \quad (18)$$

and  $J$  has zero torsion only if  $\xi_3^1 = \xi_3^2 = 0$ . In case (ii), necessarily  $x_3 \in \mathfrak{p}$  since  $\mathfrak{p}$  is abelian. Hence  $(x_3, J_{\mathfrak{p}}x_3)$  is a basis for  $\mathfrak{p}$ . Take any linear extension  $J$  of  $J_{\mathfrak{p}}$  to  $\mathfrak{n}$ . There exists some eigenvector  $y_1 \neq 0$  of  $J$  associated to some eigenvalue  $\xi_1^1 \in \mathbb{R}$ . Then  $y_1 \notin \mathfrak{p}$ , which implies  $[y_1, Jx_3] \neq 0$ , for otherwise  $y_1$  would commute to the whole of  $\mathfrak{n}$  and then be some multiple of  $x_3 \in \mathfrak{p}$ . Hence  $[y_1, Jx_3] = \lambda x_3$ ,  $\lambda \neq 0$ , and dividing  $y_1$  by  $\lambda$  one may suppose  $\lambda = 1$ . In the basis  $y_1, y_2 = Jx_3, y_3 = x_3$  one has

$$J = \begin{pmatrix} \xi_1^1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad (19)$$

and (ii) follows. □

## 7 Complex structures on $\mathfrak{n} \times \mathfrak{n}$ .

We will use for commutation relations  $[x_1, x_2] = x_5, [x_3, x_4] = x_6$ . The automorphisms of  $\mathfrak{n} \times \mathfrak{n}$  fall into 2 kinds. The first kind is comprised of the

matrices

$$\Phi = \left( \begin{array}{cccc|cc} b_1^1 & b_2^1 & 0 & 0 & 0 & 0 \\ b_1^2 & b_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3^3 & b_4^3 & 0 & 0 \\ 0 & 0 & b_3^4 & b_4^4 & 0 & 0 \\ \hline b_1^5 & b_2^5 & b_3^5 & b_4^5 & b_1^1 b_2^2 - b_2^1 b_1^2 & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & 0 & b_3^3 b_4^4 - b_4^3 b_3^4 \end{array} \right), \quad (b_1^1 b_2^2 - b_2^1 b_1^2)(b_3^3 b_4^4 - b_4^3 b_3^4) \neq 0. \quad (20)$$

The second kind ones are  $\Psi = \Theta\Phi$  where  $\Phi$  is first kind and

$$\Theta = \left( \begin{array}{cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right). \quad (21)$$

**Lemma 5.** *Any integrable complex structure  $J$  on  $\mathfrak{n} \times \mathfrak{n}$  is equivalent under some first kind automorphism to one of the following:*

$$(i) \quad \tilde{S}_\varepsilon(\xi_5^5) = \left( \begin{array}{cccccc} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_5^5 & -\varepsilon((\xi_5^5)^2 + 1) \\ 0 & 0 & 0 & 0 & \varepsilon & -\xi_5^5 \end{array} \right), \quad \varepsilon = \pm 1, \xi_5^5 \in \mathbb{R}. \quad (22)$$

$\tilde{S}_{\varepsilon'}(\xi_5'^5)$  is equivalent to  $\tilde{S}_\varepsilon(\xi_5^5)$  ( $\varepsilon, \varepsilon' = \pm 1; \xi_5'^5, \xi_5^5 \in \mathbb{R}$ ) under some first (resp. second) kind automorphism if and only if  $\varepsilon' = \varepsilon$ ,  $\xi_5'^5 = \xi_5^5$  (resp.  $\varepsilon' = -\varepsilon$ ,  $\xi_5'^5 = -\xi_5^5$ ).

$$(ii) \quad \tilde{D}(\xi_1^1) = \left( \begin{array}{cccccc} \xi_1^1 & 0 & -((\xi_1^1)^2 + 1) & 0 & 0 & 0 \\ 0 & \xi_1^1 & 0 & -((\xi_1^1)^2 + 1) & 0 & 0 \\ 1 & 0 & -\xi_1^1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\xi_1^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(\xi_1^1)^2 - 1}{2\xi_1^1} & -\frac{((\xi_1^1)^2 + 1)^2}{2\xi_1^1} \\ 0 & 0 & 0 & 0 & \frac{1}{2\xi_1^1} & \frac{1 - (\xi_1^1)^2}{2\xi_1^1} \end{array} \right), \quad \xi_1^1 \in \mathbb{R} \setminus \{0\}. \quad (23)$$

$\tilde{D}(\xi_1^1)$  is equivalent to  $\tilde{D}(\xi_1^1)$  ( $\xi_1^1, \xi_1^1 \in \mathbb{R}$ ) under some first (resp. second) kind automorphism if and only if  $\xi_1^1 = \xi_1^1$  (resp.  $\xi_1^1 = -\xi_1^1$ ).

$$(iii) \quad \tilde{T}(\xi_3^3, \xi_3^4) = \begin{pmatrix} 0 & -\xi_3^4 \xi_3^3 & -\xi_3^4 \xi_3^3 & \xi_3^4 \xi_3^3 - 1 & 0 & 0 \\ 1 & -\xi_3^3 & -\frac{(\xi_3^3)^2 + 1 - \xi_3^4 \xi_3^3}{\xi_3^3} & \xi_3^3 & 0 & 0 \\ 0 & \xi_3^3 & \xi_3^3 & -\xi_3^3 & 0 & 0 \\ 1 & 0 & \xi_3^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\xi_3^4 \xi_3^3 - 1}{\xi_3^3} & -\frac{(\xi_3^4 \xi_3^3 - 2)\xi_3^4 \xi_3^3 + (\xi_3^3)^2 + 1}{(\xi_3^3)^2} \\ 0 & 0 & 0 & 0 & 1 & \frac{\xi_3^4 \xi_3^3 - 1}{\xi_3^3} \end{pmatrix},$$

$$\xi_3^3 \in \mathbb{R} \setminus \{0\}, \xi_3^4 \in \mathbb{R}. \quad (24)$$

$\tilde{T}(\xi_3^3, \xi_3^4)$  is equivalent to  $\tilde{T}(\xi_3^3, \xi_3^4)$  ( $\xi_3^3, \xi_3^3 \in \mathbb{R} \setminus \{0\}, \xi_3^4, \xi_3^4 \in \mathbb{R}$ ) under some first (resp. second) kind automorphism if and only if  $\xi_3^3 = \xi_3^3, \xi_3^4 = \xi_3^4$  (resp.  $\xi_3^3 = -\xi_3^3, \xi_3^4 = -\xi_3^4$ ).

Finally, the cases (i), (ii), (iii) are mutually non equivalent, either under first or second kind automorphism.

*Proof.* Let  $J = (\xi_j^i)_{1 \leq i, j \leq 6}$  an integrable complex structure in the basis  $(x_k)_{1 \leq k \leq 6}$ . Denote  $J_1 = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix}$ ,  $J_2 = \begin{pmatrix} \xi_1^3 & \xi_4^1 \\ \xi_2^3 & \xi_4^2 \end{pmatrix}$ ,  $J_3 = \begin{pmatrix} \xi_1^3 & \xi_2^3 \\ \xi_4^1 & \xi_2^4 \end{pmatrix}$ ,  $J_4 = \begin{pmatrix} \xi_3^3 & \xi_4^3 \\ \xi_3^4 & \xi_4^4 \end{pmatrix}$ . Then  $J_1^* = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_5^1 \\ \xi_1^2 & \xi_2^2 & \xi_5^2 \\ \xi_1^5 & \xi_2^5 & \xi_5^5 \end{pmatrix}$  and  $J_3^* = \begin{pmatrix} \xi_1^3 & \xi_4^3 & \xi_6^3 \\ \xi_3^4 & \xi_4^4 & \xi_6^4 \\ \xi_3^6 & \xi_4^6 & \xi_6^6 \end{pmatrix}$  are zero torsion linear maps from  $\mathfrak{n}$  to  $\mathfrak{n}$ , hence equivalent to type (11), (12) or (13) in lemma 3. It can be checked that their being of different types would contradict with  $J^2 = -1$ . Hence, modulo equivalence under some first kind automorphism, we get 3 cases: case 1:  $J_1^* = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \xi_5^5 \end{pmatrix}$ ,  $J_3^* = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \xi_6^6 \end{pmatrix}$ ; case 2:  $J_1^* = D(\xi_1^1)$ ,  $J_3^* = D(\xi_3^3)$ ,  $(\xi_1^1, \xi_3^3 \neq 0)$ ; case 3:  $J_1^* = \begin{pmatrix} 0 & \xi_2^1 & 0 \\ 1 & \xi_2^2 & 0 \\ 0 & 0 & \xi_5^5 \end{pmatrix}$ ,  $J_3^* = \begin{pmatrix} \xi_3^3 & -\xi_3^3 & 0 \\ \xi_3^4 & 0 & 0 \\ 0 & 0 & \xi_6^6 \end{pmatrix}$ ,  $(\xi_2^2, \xi_3^3 \neq 0)$ . Case 1 (resp. 2, 3) leads to (22) (resp. (23), (24)) (see [1], programs "n2case1.red", "n2case2.red", "n2case3.red", and their outputs.) The assertion about equivalence in cases 1, 2 are readily proved, as is equivalence under some first kind automorphism in case 3 and the nonequivalence of the 3 types. Consider now  $\Theta \tilde{T}(\xi_3^3, \xi_3^4) \Theta^{-1}$ . It is equivalent under some first kind automorphism to some  $\tilde{T}(\eta_3^3, \eta_3^4)$ . That implies that the matrices  $\begin{pmatrix} \xi_3^3 & -\xi_3^3 \\ \xi_3^4 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & -\eta_3^4 \eta_3^3 \\ 1 & -\xi_3^3 \end{pmatrix}$  are similar, which amounts to their having same trace and same determinant, i.e.  $\eta_3^3 = -\xi_3^3, \eta_3^4 = -\xi_3^4$ . As  $\tilde{T}(\xi_3^3, \xi_3^4)$  is equivalent to  $\tilde{T}(\xi_3^3, \xi_3^4)$  under some second kind automorphism if

and only if it is equivalent to  $\Theta \tilde{T}(\xi_3^3, \xi_3^4) \Theta^{-1}$  under some first kind automorphism, the assertion about second kind equivalence in case 3 follows.  $\square$

**Remark 4.** In case 3, had we used  $J_3^* = \begin{pmatrix} 0 & \xi_4^3 & 0 \\ 1 & \xi_4^4 & 0 \\ 0 & 0 & \xi_6^6 \end{pmatrix}$ , then we would have to separate further into 2 subcases: subcase  $\xi_2^1 \neq 0$ :

$$\tilde{T}(\xi_2^1, \xi_2^2) = \begin{pmatrix} 0 & \xi_2^1 & -\frac{\xi_2^2}{\xi_2^1} & -(\xi_2^1 + 1) & 0 & 0 \\ 1 & \xi_2^2 & \frac{\xi_2^1 + 1}{\xi_2^1} & -\xi_2^2 & 0 & 0 \\ 0 & -\xi_2^1 & 0 & \xi_2^1 & 0 & 0 \\ 1 & \xi_2^2 & 1 & -\xi_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\xi_2^1 + 1}{\xi_2^2} & -\frac{(\xi_2^2)^2 + (\xi_2^1 + 1)^2}{\xi_2^2 \xi_2^1} \\ 0 & 0 & 0 & 0 & \frac{\xi_2^1}{\xi_2^2} & \frac{\xi_2^1 + 1}{\xi_2^2} \end{pmatrix}, \quad \xi_2^1 \xi_2^2 \neq 0;$$

subcase  $\xi_2^1 = 0$ :

$$\tilde{T}(\xi_2^2) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & \xi_2^2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -\xi_2^2 & -((\xi_2^2)^2 + 1) & 1 & -\xi_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\xi_2^2} & \frac{1}{\xi_2^2} \\ 0 & 0 & 0 & 0 & -\frac{(\xi_2^2)^2 + 1}{\xi_2^2} & \frac{1}{\xi_2^2} \end{pmatrix}, \quad \xi_2^2 \neq 0.$$

**Remark 5.**  $\tilde{S}_\varepsilon(\xi_5^5)$  is abelian.

**Remark 6.** If one looks for zero torsion linear maps instead of complex structures, then  $J_1^*$  and  $J_3^*$  may be of different types.

## References

- [1] <http://www.u-bourgogne.fr/monge/l.magnin/2examples/2examplesindex.html>  
or [http://www.u-bourgogne.fr/IMB/magnin/public\\_html/2examples/2examplesindex.html](http://www.u-bourgogne.fr/IMB/magnin/public_html/2examples/2examplesindex.html)
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